THE
ABEL
PRIZE
2020

The Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2020 to

Hillel Furstenberg

Hebrew University of Jerusalem, Israel

Gregory Margulis

Yale University, New Haven, CT, USA

“for pioneering the use of methods from probability and dynamics in group theory, number theory and combinatorics.”

A central branch of probability theory is the study of random walks, such as the route taken by a tourist exploring an unknown city by flipping a coin to decide between turning left or right at every cross. Hillel Furstenberg and Gregory Margulis invented similar random walk techniques to investigate the structure of linear groups, which are for instance sets of matrices closed under inverse and product. By taking products of randomly chosen matrices, one seeks to describe how the result grows and what this growth says about the structure of the group.

Furstenberg and Margulis introduced visionary and powerful concepts, solved formidable problems and discovered surprising and fruitful connections between group theory, probability theory, number theory, combinatorics and graph theory. Their work created a school of thought which has had a deep impact on many areas of mathematics and applications.

Starting from the study of random products of matrices, in 1963, Hillel Furstenberg introduced and classified a notion of fundamental importance, now called Furstenberg boundary. Using this, he gave a Poisson type formula expressing harmonic functions on a general group in terms of their boundary values. In his works on random walks at the beginning of the ’60s, some in collaboration with Harry Kesten, he also obtained an important criterion for the positivity of the largest Lyapunov exponent.

Motivated by Diophantine approximation, in 1967, Furstenberg introduced the notion of disjointness of ergodic systems, a notion akin to that of being coprime for integers. This natural notion turned out to be extremely deep and have applications to a wide range of areas including signal processing and filtering questions in electrical engineering, the geometry of fractal sets, homogeneous flows and number theory. His “×2 ×3 conjecture” is a beautifully simple example which has led to many further developments. He considered the two maps taking squares and cubes on the complex unit circle, and proved that the only closed sets invariant under both these maps are either finite or the whole circle. His conjecture states that the only invariant measures are either finite or rotationally invariant. In spite of efforts by many mathematicians, this measure classification question remains open. Classification of measures invariant by groups has blossomed into a vast field of research influencing quantum arithmetic ergodicity, translation surfaces, Margulis's version of Littlewood's conjecture and the spectacular works of Marina Ratner. Considering invariant measures in a geometric setting, Furstenberg proved in 1972 the unique ergodicity of the horocycle flow for hyperbolic surfaces, a result with many descendants.

Using ergodic theory and his multiple recurrence theorem, in 1977, Furstenberg gave a stunning new proof of Szemerédi's theorem about the existence of large arithmetic progressions in subsets of integers with positive density. In subsequent works with Yitzhak Kaztnelson, Benjamin Weiss and others, he found higher dimensional and far-reaching generalisations of Szemerédi's theorem and other applications of topological dynamics and ergodic theory to Ramsey theory and additive combinatorics. This work has influenced many later developments including the works of Ben Green, Terence Tao and Tamar Ziegler on the Hardy–Littlewood conjecture and arithmetic progressions of prime numbers.

Gregory Margulis revolutionised the study of lattices of semi-simple groups. A lattice in a group is a discrete subgroup such that the quotient has a finite volume. For semi-simple groups, Margulis classified these lattices in his "superrigidity" and "arithmeticity" theorems in the mid–1970s. Armand Borel and Harish–Chandra constructed lattices in semi-simple groups using arithmetic constructions, essentially as the group of integer-valued matrices in a large matrix group. Margulis proved that all lattices in rank 2 or higher arise from this arithmetic construction, as conjectured by Atle Selberg. In 1978, Margulis unveiled the structure of these lattices in his "normal subgroup theorem". Central to his techniques is the amazing and surprising use of probabilistic methods (random walks, Oseledets theorem, amenability, Furstenberg boundary) as well as Kazhdan property (T).

In his 1970 dissertation, Margulis constructed the so-called "Bowen–Margulis measure" of a compact Riemannian manifold of strictly negative variable curvature. Using the mixing property of geodesic flows with respect to this measure, he proved an analogue of the prime number theorem, an asymptotic formula for the number of closed geodesics shorter than a given length. Before this, the only such counting result was via the Selberg trace formula which works only for locally symmetric spaces. Since then, numerous counting and equidistribution problems have been studied using Margulis' mixing approach.

Another spectacular application of his methods is the proof in 1984 of the decades-old Oppenheim conjecture in number theory: a non-degenerate quadratic form with 3 or more variables either takes a dense set of values on the integers or is a multiple of a form with rational coeffcients.

In graph theory, Margulis's creativity resulted in his construction in 1973 of the first known explicit family of expanders, using Kazhdan property (T). An expander is a graph with high connectivity. This notion, introduced by Mark Pinsker, comes from the study of networks in communications systems. Expander graphs are now a fundamental tool in computer science and error-correcting codes. In 1988 Margulis constructed optimal expanders, now known as Ramanujan graphs, which were discovered independently by Alex Lubotzky, Peter Sarnak and Ralph Phillips.

The influence of Furstenberg and Margulis extends way beyond their results and original fields. They are recognised as pioneers by a wide community of mathematicians, from Lie theory, discrete groups and random matrices to computer science and graph theory. They have demonstrated the ubiquity of probabilistic methods and the effectiveness of crossing boundaries between separate mathematical disciplines, such as the traditional dichotomy between pure and applied mathematics.